# Linear System Theory – lecture 9

## Realizations – part 2.

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# Outline

1. SISO balanced realization.
   balancing gramians

2. Markov parameters
3. SISO balanced realization.
   balancing the obs. and ctrl. matrices.
4. MIMO, balancing gramians
5. " , " obs. and ctrl. matrices.

   5.1 realizations using the Markov parameters in the MIMO case.

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Note that both $W_c$ and $W_o$ are symmetric but $W_c \cdot W_o$ may not be symmetric.

$$W_c = Q^T D Q = \underbrace{Q^T D^{1/2}}_{R^T} \underbrace{D^{1/2} Q}_{R}$$

$$W_c > 0 \qquad D > 0$$

$$R = D^{1/2} Q$$ is not orthogonal

but is nonsingular

$$\det(\sigma^2 I - W_c W_o) =$$

$$\det(\sigma^2 I - R^T R W_o)$$

Recall that for $n \times n$ matrices $A$ and $B$

$$\det(\lambda I - AB) = \det(\lambda I - BA)$$

$$\det(\sigma^2 I - R^T W_o R)$$

which implies that $W_c W_o$ and $R W_o R^T$ have the same set of eigenvalues, thus $W_c W_o$ have positive eigenvalues.

Thm: For any $n$ dimensional state (SISO) system $(A, b, c)$, there exists an equivalence transformation

$$\bar{x} = Px$$

such that the controllability gramian $\overline{W_c}$ and observability gramian $\overline{W_o}$ of its equivalent system have the property

$$\boxed{\overline{W_c} = \overline{W_o} = \Sigma_1 \quad \text{(diagonal)} \\ \text{This is called a balanced realization.}}$$

Proof:

$$W_c = R^T R$$

can be diagonalized

$$R W_o R^T \stackrel{\checkmark}{=} U \Sigma_1^2 U^T$$

$U$ is orthogonal, i.e. $U^T U = I$

let

$$\boxed{\begin{aligned} P^{-1} &\triangleq R^T U \Sigma_1^{-1/2} \\ P &= \Sigma_1^{1/2} U^T (R^T)^{-1} \end{aligned}} \quad (*)$$

then from $W_c = R^T R$

one gets, after setting $P$ in ⊛ in

$$\bar{W}_c = P W_c P^T$$

$$\bar{W}_c = \Sigma_1^{1/2} U^T (R^T)^{-1} \underbrace{W_c}_{R^T R} R^{-1} U \Sigma_1^{1/2}$$

$$= \Sigma_1$$

and from $R W_o R^T = U \Sigma_1^2 U^T$

$$\bar{W}_o = (P^T)^{-1} W_o P^{-1}$$

$$(P^T)^{-1} = (P^{-1})^T$$

$$\bar{W}_o = \Sigma_1^{-1/2} U^T R W_o R^T U \Sigma_1^{-1/2}$$

$$= \Sigma_1^{-1/2} U^T U \Sigma_1^2 U^T U \Sigma_1^{-1/2}$$

$$= \Sigma_1^{-1/2} \Sigma_1^2 \Sigma_1^{-1/2} = \Sigma_1$$

$$\Sigma_1 = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$$

the product $W_c W_o$ of any minimal realization is similar to $\Sigma_o$.

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$\gamma$

2. Markov parameters (SISO case)

Recall the Markov parameters:

$$CB, \quad CAB, \quad CA^2B, \quad \dots$$

2.1 Realization from Markov parameters

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_{n-1} s + \alpha_n}$$

$$G(s) = h(0) + h(1) s^{-1} + h(2) s^{-2} + \dots +$$

$$h(m) = \frac{d^{m-1}}{dt^{m-1}} g(t)$$ ← impractical due to noise.

$g(t)$ is the impulse response

$$\beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_n = (s^n + \alpha_1 s^{n-1} + \dots + \alpha_n)(h(0) + h(1) s^{-1} + h(2) s^{-2} + \dots)$$

$$h(1) = \beta_1$$

$$h(2) = -\alpha_1 h(1) + \beta_2$$

$$h(3) = -\alpha_2 h(2) - \alpha_1(1) h(1) + \beta_3$$

$$\vdots$$

$$h(n) = -\alpha_1 h(n-1) - \alpha_2 h(n-2) - \dots - \alpha_{n-1} h(n) + \beta_n$$

## Hankel Matrix

$$T(\alpha, \beta) = \begin{bmatrix} h(1) & h(2) & h(3) & \cdots & h(\beta) \\ h(2) & h(3) & h(4) & \cdots & h(\beta+1) \\ h(3) & & & & \vdots \\ \vdots & & & & \\ h(\alpha) & h(\alpha+1) & & \cdots & h(\alpha+\beta-1) \end{bmatrix}$$

**Theorem:** a strictly proper rational function $G(s)$ has degree $n$, if and only if,

$$\operatorname{rank}(T(n,n)) = \operatorname{rank}(T(n+k, n+l)) = n$$

for any $k, l = 1, 2, \dots$

Proof:

$$h(m) = -\alpha_1 h(m-1) - \alpha_2 h(m-2) - \dots - \alpha_{n-1} h(m-n+1) - \alpha_n h(m-n) \quad (*)$$

$$(**) \quad \text{rank}(T(n,n)) = \text{rank}(T(n+1,n)) = \text{rank}(T(\infty,n))$$

because (*) $n+1$ rows can be expressed by the previous rows, therefore (**) holds

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now $g(s) = h(1) s^{-1} + h(2) s^{-2} + \dots$

(remark strictly proper implies $h(0) = 0$)

then using (**) we can compute $\alpha_i$ of (*) and we use

$$h(1) = \beta_1$$
$$h(2) = -\alpha_1 h(1) + \beta_2$$
$$h(3) = -\alpha_1 h(2) - \alpha_2 h(1) + \beta_3$$
$$\vdots$$
$$h(n) = \quad \dots -$$

so as to determine the $\beta_i$

# Realization problem: (strictly proper SISO)

$$G(s) = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3}$$

$$= C(sI-A)^{-1}B$$

$(A, B, C)$ is a realization

$$\Longleftrightarrow$$

$$h(1) = CB \qquad h(m) = CA^{m-1}B$$
$$h(2) = CAB$$
$$h(3) = CA^2B$$

$$T(n,n) = \begin{bmatrix} CB & CAB & CA^2B \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & \vdots \\ CA^2B & \cdots & & \\ \vdots & & & \\ CA^{n-1}B & \cdots & & CA^{2(n-1)}B \end{bmatrix}$$

which implies

$$T(n,n) = \mathcal{O}\,\mathcal{C}$$

Define

$$\tilde{T}(n,n) = \begin{array}{cc} \overset{n}{\longleftrightarrow} & \\ \begin{bmatrix} h(2) & h(3) & \cdots & h(n+1) \\ h(3) & & & \vdots \\ h(4) & & & \\ & & & \\ h(n+1) & \cdots & & h(2n) \end{bmatrix} & \updownarrow n \end{array}$$

$$\boxed{\begin{aligned} \tilde{T}(n,n) &= \mathcal{O} A \mathcal{C} \\ T(n,n) &= \mathcal{O}\mathcal{C} \end{aligned}}$$

We can obtain many different realizations.
The simplest is to select $\mathcal{O} = I$

$$\mathcal{C} = T(n,n)$$

$$A = \tilde{T}(n,n)\, T^{-1}(n,n)$$

The first row of $\mathcal{O} = I$ gives

$$C = [\,1 \quad 0 \quad 0 \;\cdots\; 0\,]$$

$$\left(\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right)$$

The first column of $T(n,n)$ gives $B$

$$B = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(n) \end{bmatrix}$$

We also have

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \ddots & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} T(n,n) = \tilde{T}(n,n)$$

We get an observable canonical form by forcing
$$\mathcal{O} = I.$$

$$G(s) = \frac{4s^2 - 2s - 6}{2s^4 + 2s^3 + 2s^2 + 3s + 1}$$

$$0 \cdot s^{-1} + 2s^{-2} - 3s^{-3} - \underbrace{2s^{-4} + 2s^{-5}}_{T} + \underbrace{3{,}5s^{-6}}_{\tilde{T}} + \cdots$$

1°/ We constitute $T(4,4)$
and compute its rank, it is 3.

$\Longrightarrow$ numerator and denominator have a common factor, not coprime

$$\tilde{T}(3,3) = \begin{bmatrix} 2 & -3 & -2 \\ -3 & 2 & 2 \\ -2 & 2 & 3,5 \end{bmatrix}$$

$$T(3,3) = \begin{bmatrix} 0 & 2 & -3 \\ 2 & -3 & -2 \\ -3 & -2 & 2 \end{bmatrix}$$

$$A = \tilde{T}\,T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -1 & 0 \end{bmatrix}$$

$$\mathcal{O} = I$$

$$\hookrightarrow \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

## Balanced form SISO case.

A realization with the property

$$\mathcal{C}\mathcal{C}^T = \mathcal{O}^T\mathcal{O}$$

( this is different from balancing the gramians)

We use singular value decomposition

(recall lin. alg. : $M \updownarrow^m_{\overset{\longleftrightarrow}{n}} = K \Sigma_1 L^T$

not necess. square

$A$ is arbitrary, $K$ and $L$ are orthogonal

$$K \cdot K^T = I_m \qquad L L^T = I_n$$

and $\Sigma_1 = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \sigma_3 \end{bmatrix}$ $\quad \underline{\underline{\sigma_i > 0}}$

svd in matlab

$$T(n,n) \overset{\text{svd}}{=} K \Lambda L^T = \underbrace{K \Lambda^{1/2}}_{\mathcal{O}} \underbrace{\Lambda^{1/2} L^T}_{\mathcal{C}}$$

$$T = \mathcal{O} \cdot \mathcal{C}$$

$$\mathcal{C} \triangleq \Lambda^{1/2} L^T$$

$$\mathcal{O} \triangleq K \Lambda^{1/2}$$

$$\mathcal{O}^{-1} = \Lambda^{-1/2} K^T \qquad \mathcal{C}^{-1} = L \Lambda^{-1/2}$$

$$\tilde{T} = \mathcal{O} A \mathcal{C}$$

$$\boxed{\begin{array}{ll} A = \mathcal{O}^{-1} \tilde{T} \mathcal{C}^{-1} & \\ B & \text{first column of } \mathcal{C} \\ & \text{(since } \mathcal{C} = [B \ AB \cdots A^{n-1}B] \text{ by definition)} \\ C & \text{first row of } \mathcal{O} \\ & \text{(since } \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^{n-1} \end{bmatrix} \text{)} \end{array}}$$

is a minimal realization and balanced.

$$\mathcal{C}\mathcal{C}^T = \Lambda^{1/2} L^T L \Lambda^{1/2} = \Lambda$$

$$\mathcal{O}^T \mathcal{O} = \Lambda^{1/2} K^T K \Lambda^{1/2} = \Lambda$$

4. MIMO balanced realizations

balancing the gramians.

→ works exactly as for the SISO case

→ we need however to show that two minimal realizations are similar.

5. MIMO balanced realizations

balancing obs and ctrl. matrices.

scalar case (SISO)

$$\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$$

$$P = \bar{\mathcal{O}}^{-1}\mathcal{O}$$

$$= \bar{\mathcal{C}}\mathcal{C}^{-1}$$

matrix case (MIMO)

$\mathcal{O}$ and $\bar{\mathcal{O}}$ $nq \times n$ of rank $n$.

$\mathcal{C}$ and $\bar{\mathcal{C}}$ $n \times np$ of rank $n$.

$$\mathcal{O} A \mathcal{C} = \tilde{T}$$

Let us introduce the pseudo-inverses $\mathcal{O}^+$ and $\mathcal{C}^+$:

$$\mathcal{O}^+ \triangleq (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T$$

$$\mathcal{O}^+\mathcal{O} = I$$

$$\mathcal{C}^+ \triangleq \mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}$$

$$\mathcal{C}\mathcal{C}^+ = I$$

$A \triangleq \mathcal{O}^+\tilde{T}\mathcal{C}^+$

$B$ first $p$ columns of $\mathcal{C}$ (by def. of $\mathcal{C} = [B \; AB \; \cdots]$)

$C$ first $q$ rows of $\mathcal{O}$ (by def. $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$)

$$\bar{\mathcal{C}} = (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T \cdot \mathcal{O}\mathcal{C} = P\mathcal{C}$$

$$\bar{\mathcal{O}} = \mathcal{O}\mathcal{C}\bar{\mathcal{C}}^T(\bar{\mathcal{C}}\bar{\mathcal{C}}^T)^{-1} = \mathcal{O}P^{-1}$$

$$P = \bar{\mathcal{O}}^+\mathcal{O} = (\bar{\mathcal{O}}^T\bar{\mathcal{O}})^{-1}\bar{\mathcal{O}}^T\mathcal{O}$$
$$= \bar{\mathcal{C}}\mathcal{C}^+ = \bar{\mathcal{C}}\mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}$$

3.1 Realization from the Markov parameters (in the MIMO case) strictly proper

$$G(s) = \underset{q}{\overset{p}{H}}(1)\, s^{-1} + H(2)\, s^{-2} + H(3)\, s^{-3} + \dots$$

$$\underset{q}{\overset{p}{CB}}$$

$$T = \begin{bmatrix} H(1) & H(2) & H(3) & \dots \\ H(2) & & & \\ H(3) & & & \\ \vdots & & & \\ H(r) & & & \end{bmatrix}$$

$$\tilde{T} = \begin{bmatrix} H(2) & H(3) & H(4) & \dots \\ H(3) & & & \\ H(4) & & & \\ \vdots & & & \\ H(r+1) & & & \end{bmatrix}$$

$$\boxed{\begin{aligned} T &= \mathcal{O}\,\mathcal{C} \\ \tilde{T} &= \mathcal{O} A\, \mathcal{C} \end{aligned}}$$

$rq \times n$

$n \times rp$

When the rank becomes constant $r$ is equal to the degree of the minimal polynomial

$$T = K \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} L^T$$

$$\Lambda = \mathrm{diag}(\lambda_i)$$

$$\lambda_1, \ldots, \lambda_n$$

clearly $n$ is the rank of $T$.

let $\bar{K}$ denote the first $n$ columns of $K$.

and $\bar{L}^T$ denote the first $n$ rows of $L^T$

$$T = \bar{K} \Lambda \bar{L}^T = \underline{\bar{K} \Lambda^{1/2}} \, \underline{\underline{\Lambda^{1/2} \bar{L}^T}}$$

$$= \underline{\mathcal{O}} \, \underline{\underline{\mathcal{C}}}$$

$$\underline{\mathcal{O} = \bar{K} \Lambda^{1/2}} \quad \text{and} \quad \underline{\underline{\mathcal{C} = \Lambda^{1/2} \bar{L}^T}}$$

Note that $\mathcal{O}$ is $n_y \times n$

$\mathcal{C}$ is $n \times n_p$

$$\mathcal{O}^+ = \left[\Lambda^{1/2}\bar{K}^T\bar{K}\Lambda^{1/2}\right]^{-1}(\Lambda^{1/2})^T\bar{K}^T$$

$$\mathcal{O}^+ = \Lambda^{1/2}\bar{K}^T$$

$$\mathcal{C}^+ = \bar{L}\Lambda^{1/2}$$

$A = \mathcal{O}^+\tilde{T}\mathcal{C}^+$

$B$ first $p$ columns of $\mathcal{C}$

$C$ first $q$ rows of $\mathcal{O}$

$$\mathcal{O}^T\mathcal{O} = \Lambda^{1/2}\bar{K}^T\bar{K}\Lambda^{1/2} = \Lambda$$

$$\mathcal{C}\mathcal{C}^T = \Lambda^{1/2}\bar{L}^T\bar{L}\Lambda^{1/2} = \Lambda$$